Low energy theorem for virtual Compton scattering and generalized sum rules of the nucleon

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We formulate the low energy theorem for virtual Compton scattering off a nucleon and examine its consequences for generalized nucleon polarizabilites. As a result of a new, model independent definition of the low energy limit for VVCS reaction, all generalized sum rules of the nucleon have continuous limit for real photons and obtain contributions from the t-channel that were not included previously.

PACS numbers: 11.55.Hx, 14.20.Dh, 13.60.Fz, 13.60.Hb

Understanding proton structure remains to be one of the top priorities in physics of strong interactions. The quest started in the 30's with the first measurement of magnetic moment of the nucleon [1] that was found to be anomalously large indicating that proton is not a point-like Dirac particle. In the late 50's, application of dispersion relations to real Compton scattering together with a low energy theorem (LET) gave rise to nucleon sum rules. The sum rules express a model-independent correspondence between static properties of the nucleon and photo-absorption spectrum [2]. In particular, the Gerasimov-Drell-Hearn (GDH) sum rule relates unambiguously the value of the anomalous magnetic moment (a.m.m.) of the nucleon κ to the spin-dependent cross-section [3], $-e^2\kappa^2/2M^2 = (1/\pi)\int d\nu(\sigma_{1/2} - \sigma_{3/2})/\nu$, with e the electric charge, M the nucleon mass, and the helicity cross sections $\sigma_{1/2}(\sigma_{3/2})$ refer to total photo-absorption $\gamma + N \to X$ of circularly polarized photons by polarized nucleon with the helicities parallel (antiparallel) to each other, respectively. The validity of the GDH sum rule was proven analytically in QED for the electron that receives a non-zero a.m.m., $\alpha_{em}/2\pi$ from radiative corrections [4]. Unfortunately, such a consistency check within QCD is not possible, and it is important to have a direct comparison of photo-absobtion nucleon sum rule with the experimental data. Such data were obtained by the GDH collaboration at MAMI [5] that confirmed its validity within experimental precision.

A more detailed study of the internal structure of the nucleon has become possible with electron scattering experiments where the electromagnetic interaction is mediated by a virtual photon. Elastic electron scattering probes spatial distribution of charge and magnetization inside the nucleon [6]. Further insight into nucleon structure came from deep inelastic electron scattering (DIS) experiments at SLAC [7] that confirmed that quarks and gluons are the building blocks of the nucleon. Having observed the intimate connection between static properties of the nucleon and the photo-absorption cross section, it is logical to expect a similar relation between form factors and absorption cross section for virtual photons.

In [8], an attempt to obtain such generalized sum rules was made by considering the forward limit of symmetric virtual Compton scattering (VVCS) with equal virtualities of the two (spacelike) photons $Q^2 = -q^2 = -q'^2$. Applying analyticity and unitarity to the four independent amplitudes describing this process, their imaginary parts were related to the DIS structure functions. Their real parts are obtained from dispersion relations at fixed Q^2 and fixed $t = 0$. According to the idea of LET, these real parts are expanded into Taylor series in variable ν, thus reducing the problem to a dispersion representation of the low energy coefficients of the expansion (constants in ν and functions of Q^2). The lowest order coefficients, on the other hand, can be calculated directly from the nucleon-pole graphs and thus related to the elastic form factors. However, in [8], these constants were obtained at such a kinematical point where the virtual photon brings the nucleon off its mass shell, and where no information on the form factors is available. As a result, the low energy limit of the spin-independent VVCS amplitude does not match with that for real photons, the well-known Thomson term. The situation is even worse for the spin-dependent part: if taken at finite Q^2, its low energy limit according to [8] has a singularity at the real photon point. Correspondingly, the GDH sum rule as function of Q^2 is in general not defined and only exists for real photons, and the only generalization that is achieved concerns the GDH integral.

The main goal of this letter is to show how these problems are circumvented when low-energy expansion is set up around the correct point in the Mandelstam plane of kinematical variables and derive a new set of sum rules for the low energy parameters.

The twelve invariant amplitudes describing symmetric VVCS are functions of three independent Mandelstam variables, which are conveniently chosen as $\nu = (s-u)/4M$, t, and Q^2 where, $s = (p+q)^2$, $u = (p-q')^2$, $t = (p'-p)^2$ satisfy $s+t+u+2Q^2 = 2M^2$ with M being the nucleon mass. Since the s channel, $\gamma^*(q) + N(p) \to \gamma^*(q') + N(p')$, and u channel, $(q \leftrightarrow -q')$ represent the same physical process, the individual amplitudes have

well defined parity under $\nu \to -\nu$ exchange, which makes the variable ν particularly useful. For fixed Q^2, the physical region of the s-channel reaction is fixed by the following conditions: *i)* the total energy is above the threshold for the reaction in the s-channel, while *ii)* it is below physical thresholds in u- and t-channels, and *iii)* the cosine of the scattering angle takes physical values, and similarly for the u-channel. The Mandelstam $\nu - t$ plane is shown in Fig. 1. The t-channel physical region lies at

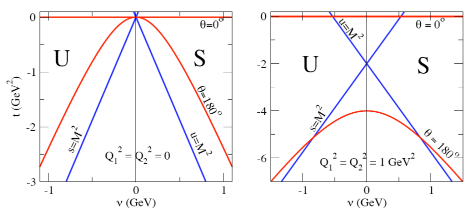

FIG. 1: The Mandelstam ν, t plane of real Compton scattering (RCS) (left panel) and symmetric VVCS (right panel). The physical region is bound by the $\theta = 0^0, 180^0$ lines. The nucleon pole in the Born s (u) channel amplitudes is located along the $s(u) = M^2$ lines. The $t = 0$ and $\theta = 0^0$ lines coincide.

$t \geq 4M^2$ and is not shown here.

We next consider the analytical structure of a VVCS amplitude. It has two distinct classes of contributions: the (generalized) Born contributions that correspond to a single-particle exchange in the s, u or t-channel, shown in Fig.2, and the remaining, non-Born contributions that

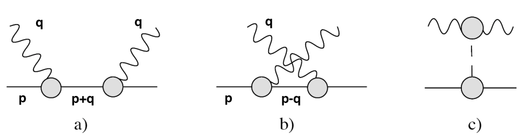

FIG. 2: The generalized Born contributions to Compton scattering: nucleon exchange diagrams (a,b) and the t-channel π^0-exchange (c). The blobs denote form factors.

are associated with two or more particle intermediate states in either channel. These two contributions have different analytical structure, so they should be treated separately. Furthermore, away from the physical nucleon point, $s = u = M^2$, the Born amplitude, however, contains off-mass shell particles and have to be treated with care to avoid model dependence. For the general case $Q_1^2 \neq Q_2^2$ we can summarize these observations in the following statement: *Low Energy Theorem for VVCS: The low energy expansion for the general VVCS reaction is defined in the Mandelstam ν, t plane around the $\nu = 0,\ q \cdot q' = -(t + Q_1^2 + Q_2^2)/2 = 0$ point which automatically corresponds to $s = u = M^2$. At this point, the contribution from the nucleon-exchange, Born amplitude is unambiguous given in terms of physical observables and it can be isolated. The generalized polarizabilities parametrize the leading terms of the low energy expansion of the rest amplitude in powers of ν.*

Note that in Breit frame defined by $\vec{p} + \vec{p'} = 0$ the low energy limit defined above corresponds to vanishing of both photon energies since $\omega = \omega' = M\nu/\sqrt{M^2 - t/4}$, independently on the virtualities. In contrast, in [8], the low-energy expansion was formulated as expansion in ν around the point $\nu = 0, t = 0$ instead, and this lead to the controversial results discussed earlier. In the following, we will use the $\nu = q \cdot q' = 0$ point to define the low energy expansion and rederive sum rules for nucleon polarizabilities. We will demonstrate that once the correct point for low energy expansion is chosen, the generalized Thomson term is obtained correctly *i.e.* the VVCS amplitude reduces to that for RCS in the limit $Q^2 \to 0$. To use the power of dispersion relations in VVCS it is necessary to introduce a set of scalar amplitudes, $F_i(\nu, t, Q^2)$ defined as coefficients in expansion of the hardonic tensor $T^{\mu\nu}$ in a basis of twelve independent tensors $T_i^{\mu\nu}$ that are gauge-invariant and free from kinematical singularities and constraints, [9, 10],

$$\bar{u}(p')T^{\mu\nu}u(p) = \sum_i \bar{u}(p')T_i^{\mu\nu}u(p)F_i(\nu, t, Q^2) \qquad (1)$$

In the above sum the label i takes on twelve values i = 1,2,4,6,7,8,10,12,17,18,19,21 - using the original labels from [9, 10] where the general ($Q_1^2 \neq Q_2^2$) VVCS amplitude was considered. In the forward limit, $t = 0$, only four out of the twelve tensors are independent and can be chosen to coincide with $T_i^{\mu\nu}$ for i = 1,2,7,17. From the remaining eight tensors j = 4,6,8,10,12,18,19,21 $\neq i$ we extract the components $\tilde{T}_j^{\mu\nu}$ that vanish in the forward direction, $\tilde{T}_j^{\mu\nu} = T_j^{\mu\nu} - \sum_{i=1,2,7,17 \neq j} c_{ij} T_i^{\mu\nu}$ and write the VVCS amplitude as $T^{\mu\nu} = \sum_i f_i T_i^{\mu\nu} + \sum_j F_j \tilde{T}_j^{\mu\nu}$. The first term, with $f_i = F_i + \sum_j c_{ij} F_j$ contains the four tensors that are independent in the forward limit and the second term contains the remaining eight tensors that complete the basis for $t \neq 0$ and vanish in the forward limit. Our four forward VVCS amplitudes $f_i = f_i(\nu, t, Q^2)$ are related to the amplitudes $T_{1,2}, S_{1,2}$ of [8] by

$$\begin{aligned}
\hat{f}_1 &\equiv f_1 = -\frac{1}{2M}\frac{1}{Q^2}\left[T_1 - \frac{1}{2x}T_2\right] \\
\hat{f}_2 &\equiv \nu^2 f_1 = \frac{1}{2M}\frac{1}{4M^2}\frac{1}{2x}T_2 \\
\hat{f}_7 &\equiv \nu f_7 = \frac{1}{8M^3}S_1 \\
\hat{f}_{17} &\equiv \nu f_{17} = -\frac{\nu}{4M^3}S_2
\end{aligned} \qquad (2)$$

As discussed earlier, s and u channel nucleon exchange is unambiguous at the low energy, $\nu = 0, t = -2Q^2$ point

and given by $\hat{f}_i^B(Q^2) = \lim_{\nu\to 0}\hat{f}_i^B(\nu, t=-2Q^2, Q^2)$,

$$\begin{aligned}
\hat{f}_1^B(Q^2) &= -\frac{e^2}{4M^3}F_P^2 \qquad (3)\\
\hat{f}_2^B(Q^2) &= -\frac{e^2}{4M^3}F_D^2\\
\hat{f}_7^B(Q^2) &= -\frac{1}{8M^3}F_P^2\\
\hat{f}_{17}^B(Q^2) &= \frac{1}{4M^2}\left[(F_D+F_P)^2 + \frac{Q^2}{8M^2}F_P^2\right]
\end{aligned}$$

where $F_{D,P}(Q^2)$ stand for the Dirac and Pauli elastic form factors of the nucleon (instead of the usual notation $F_{1,2}$ to avoid confusion with the DIS structure functions). Note that residues of the poles in the s and u channel vanish at $t=-2Q^2$ so $\hat{f}_i^B$ are finite. The ν^{-1}, ν^{-2} singularities in the Born amplitudes in Eq.(3) f_2^B, f_7^B, f_{17}^B are cancelled in the full amplitude $T^{\mu\nu}$ by corresponding zeros in $T_i^{\mu\nu}$, for $i=2,7,17$. Finally, we notice for completeness that the pion-pole of Fig. 2, (c) does not contribute to any of the four amplitudes. It is instructive to compare these results with the well-known results for real Compton scattering. The spin-independent real Compton amplitude is known to be finite at zero energy and given by the classical Thomson term $\bar{u}(p)T^{\mu\nu}(\nu=0, t=0, Q^2=0)u(p) = 2e^2 g^{\mu\nu}$ [13]. Using the tensors $T_i^{\mu\nu}$ from [9, 10] and picking the term $\sim g^{\mu\nu}$, we obtain the correct answer when taking $Q^2=0$ limit.

The amplitudes $\hat{f}_i$ defined above are free from kinematical singularities and can be used in dispersion relation calculation. All four amplitudes $\hat{f}_i$ are even functions of ν and as a result obey a dispersion relation at fixed-t and Q^2 in the form

$$\mathrm{Re}\hat{f}_i(\nu,t,Q^2) = \frac{1}{\pi}\int_{\nu_0^2}^{\infty}\frac{d\nu'^2}{\nu'^2-\nu^2}\mathrm{Im}\hat{f}_i(\nu',t,Q^2), \qquad (4)$$

unless a subtraction is needed to ensure the convergence of the integral. In the forward direction, $t=0$, the optical theorem relates imaginary parts to the DIS structure functions,

$$\begin{aligned}
\mathrm{Im}\hat{f}_1(\nu,0,Q^2) &= -\frac{\pi e^2}{MQ^2}\left[F_1(x,Q^2) - \frac{1}{2x}F_2(x,Q^2)\right]\\
\mathrm{Im}\hat{f}_2(\nu,0,Q^2) &= \frac{\pi e^2}{4M^3}\frac{1}{2x}F_2(x,Q^2)\\
\mathrm{Im}\hat{f}_7(\nu,0,Q^2) &= \frac{\pi e^2}{4M^2\nu}g_1(x,Q^2)\\
\mathrm{Im}\hat{f}_{17}(\nu,0,Q^2) &= -\frac{\pi e^2}{2M\nu}g_2(x,Q^2), \qquad (5)
\end{aligned}$$

where the usual Bjorken variable $x = Q^2/2M\nu$ was introduced. To obtain the generalized sum rules, the above forward dispersion relations have to be connected to the $t=-2Q^2$ point where the separation of the VVCS amplitude into Born and non-Born parts is well defined. This can be done by invoking analyticity in the t-channel and it leads to subtracted dispersion relations in t for each of the four amplitudes,

$$\begin{aligned}
&\mathrm{Re}\hat{f}_i(\nu,-2Q^2,Q^2) - \mathrm{Re}\hat{f}_i(\nu,0,Q^2) \qquad (6)\\
&= -\frac{2Q^2}{\pi}\int\frac{dt'}{t'(t'+2Q^2)}\mathrm{Im}\hat{f}_i(\nu,t',Q^2)
\end{aligned}$$

This t-channel contribution does not affect the real photon point since for real photons the low energy expansion is defined at $t=0$ but it is non-zero at finite Q^2. Finally, we expand the unknown non-Born part of the amplitudes around the point $\nu=0$ at $t=-2Q^2$. To accomplish this, we rewrite the VVCS tensors in terms of electric and magnetic fields of the initial and final photon. We next define the field strength tensors , $F^{\mu\nu} = -ie(q^\mu\epsilon^\nu - q^\nu\epsilon^\mu)$ and $F'^{\mu\nu} = -ie(q'^\mu\epsilon'^\nu - q'^\nu\epsilon'^\mu)$. As discussed earlier, it is the Breit frame, where energies of the initial and final photon vanish simultaneously at the nucleon point. Then, the spin-independent tensors can be rewritten through the electromagnetic fields in Breit frame as

$$\begin{aligned}
e^2\varepsilon_\mu\varepsilon_\nu'^{*}T_1^{\mu\nu} &= -\frac{1}{2}F^{\mu\nu}F'^{*}_{\mu\nu} = \vec{E}\cdot\vec{E}'^{*} - \vec{B}\cdot\vec{B}'^{*} \qquad (7)\\
e^2\varepsilon_\mu\varepsilon_\nu'^{*}T_2^{\mu\nu} &= -4(P_\mu F^{\mu\alpha})(P^\nu F'^{*}_{\nu\alpha}) = (4M^2-t)\vec{E}\cdot\vec{E}'^{*}
\end{aligned}$$

The dipole polarizabilities quantify the electric (magnetic) dipole induced in low-energy external field $\vec{E}(\vec{B})$ in the direction of the field, $\vec{d} = 4\pi\alpha\vec{E}$, $\vec{m} = 4\pi\beta\vec{B}$. These dipoles then interact with the outgoing photon electric and magnetic fields, respectively. Approximating the response of the internal structure of the nucleon by the linear form, we arrive to the low energy expansion of the spin-independent amplitudes, $\hat{f}_{1,2}(\nu,-2Q^2,Q^2)$,

$$\begin{aligned}
\hat{f}_1 &= -\frac{e^2}{4M^3}F_P^2 - 4\pi\beta(Q^2) + O(\nu^2) \qquad (8)\\
\hat{f}_2 &= -\frac{e^2}{4M^3}F_D^2 + \frac{4\pi\nu^2}{4M^2+2Q^2}\left[\alpha(Q^2)+\beta(Q^2)\right]\\
&+ O(\nu^4)
\end{aligned}$$

We notice that for low energy virtual photons in Breit frame, only the transverse components are small, $\vec{E}_T = \omega\vec{\varepsilon}_T = O(\nu)$, while the longitudinal electric and (transverse) magnetic components are finite, $E_L, B \sim |\vec{q}| \sim \sqrt{Q^2}$. Because of the choice of the kinematical point $t=-2Q^2$, however, the longitudinal electric fields in the spin-independent part contribute at order $O(\nu^2)$: $(\vec{E}_L\cdot\vec{E}_L'^{*}) \sim (\vec{q}\cdot\vec{q}') = \omega^2 = O(\nu^2)$. This explains, why contribution from polarizabilities to $\hat{f}_2$ is of relative order ν^2 with respect to the Born contribution, while the contribution of the magnetic polarizability to $\hat{f}_1$ appears at the same order in energy as the Born contribution. The low energy expansion of Eq.(8) is a new result. We contrast this with the result of [8] where the low energy

expansion of the amplitudes $f_{1,2}$ in the forward direction is in terms of the polarizabilities α_L and $(\alpha+\beta)$ instead of β and $(\alpha+\beta)$, respectively with both expansions starting at relative order ν^2 obtained by a mere analogy with real Compton scattering. Our results demonstrate that this analogy may be misleading.

Finally we write down the generalized sum rules for the low energy constants appearing the expansion of all four amplitudes $\hat{f}_{1,2,7,17}$. Using Eqs. 2,4, 7 for the first three amplitude we find,

$$\begin{aligned}
\frac{e^2}{4M^3}F_P^2 + 4\pi\beta(Q^2) &= \frac{2e^2}{MQ^2}\int_0^{x_\pi}\frac{dx}{x}\left[F_1(x,Q^2) - \frac{1}{2x}F_2(x,Q^2)\right] + \frac{2Q^2}{\pi}\int\frac{dt'}{t'(t'+2Q^2)}\mathrm{Im}\hat{f}_1(0,t',Q^2)\\
4\pi\left[\alpha(Q^2)+\beta(Q^2)\right] &= \frac{4e^2M}{Q^4}\left[1+\frac{Q^2}{2M^2}\right]\int_0^{x_\pi}dxF_2(x,Q^2) - \frac{2Q^2}{\pi}\int dt'\frac{1}{t'(t'+2Q^2)}\frac{d}{d\nu^2}\mathrm{Im}\hat{f}_1(0,t',Q^2)\\
-\frac{e^2}{8M^3}F_P^2(Q^2) &= \frac{e^2}{MQ^2}\int_0^{x_\pi}dxg_1(x,Q^2) - \frac{2Q^2}{\pi}\int\frac{dt'}{t'(t'+2Q^2)}\mathrm{Im}\hat{f}_7(0,t',Q^2)
\end{aligned} \tag{9}$$

where the upper limit of integration x_π corresponds to the threshold of pion production, $x_\pi = Q^2/(Q^2+m_\pi^2+2Mm_\pi) < 1$. The integrals over t receive contributions from unitarity in the t-channel starting with the reaction $\gamma\gamma\to\pi\pi\to N\bar{N}$ at $t\geq(2m_\pi)^2$, and from the su spectral region at negative t. The sum rule for the amplitude $\hat{f}_{17}$ gives rise to the Burkhard-Cotingham sum rule, $\int_0^1 g_2(x,Q^2)dx = 0$. This sum rule is not related to the low energy expansion of the corresponding amplitude, and that is why we dropped it here.

The first sum rule that give a dispersion relation representation for the magnetic polarizability β is new. Due to the structure of the tensor T_1, $T_1 = -(qq')g^{\mu\nu} + q^\nu q'^\mu$ at $(qq')=0$, the result is purely magnetic, whereas at $t=0$ it has a longitudinal component that led the authors of [8] to the sum rule for α_L. The convergence of this sum rule is however unclear. The second sum rule is the generalization of Baldin sum rule to which it reduces at $Q^2=0$. As compared to the corresponding generalization from [8] we notice (besides the t-channel piece) the relative factor of $1+Q^2/2M^2$ whose origin is in the usual Breit frame factor $(p^0)^2 = M^2 - t/4$ entering the low energy expansion of $\hat{f}_2$ instead of M^2 in the lab frame. The third sum rule is the generalization of the GDH sum rule. At the real photon point it has a negative value, whereas the high Q^2 data for the first moment of g_1 suggest a small positive value. The transition between the two regimes is a subject of intensive experimentally studies [14]. Our results suggest that at finite Q^2 the sum rule has the t-channel contribution that is missing in previous analyses. Due to small pion mass, it is expected to lead to a fast variation of this contribution at low Q^2. The indications of this behaviour were obtained in HBChPT [11] although the range of validity of those calculations is constrained to very low values of Q^2. We leave the computation of the t-channel contribution to an upcoming work. This work was supported in part by the US Department of Energy grant under contract DE-FG0287ER40365, by the US National Science Foundation under grant PHY-0555232.

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